

Univalence criterion for meromorphic functions and Loewner chains

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Dedicated to Professor H. M. Srivastava on the Occasion of his Seventieth Birth Anniversary

Abstract. The object of the present paper is to obtain a more general condition for univalence of meromorphic functions in the \mathbb{U}^* . The significant relationships and relevance with other results are also given. A number of known univalent conditions would follow upon specializing the parameters involved in our main results.

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1. INTRODUCTION

We denote by \mathbb{U}_r the disk $\{z \in \mathbb{C} : |z| < r\}$, where $0 < r \leq 1$, by $\mathbb{U} = \mathbb{U}_1$ the open unit disk of the complex plane and $\mathbb{U}^* = \mathbb{C} \setminus \overline{\mathbb{U}}$, where $\overline{\mathbb{U}}$ is closure of \mathbb{U} .

Let \mathcal{A} denote the class of all analytic functions in the open unit disk \mathbb{U} normalized by

$$f(z) = z + a_2 z^2 + \dots \quad (z \in \mathbb{U})$$

and we denote by \mathcal{S} the subclass of \mathcal{A} consisting of functions which are also univalent in \mathbb{U} . Closely related to \mathcal{S} is the class Σ of all meromorphic functions in \mathbb{U}^* by

$$f(\zeta) = b\zeta + b_0 + \frac{b_1}{\zeta} + \dots \quad (\zeta \in \mathbb{U}^*)$$

and Σ_0 stands for all functions from Σ with normalization $b = 1$ and $b_0 = 0$. These classes have been one of the important subjects of research in complex analysis especially, Geometric Function Theory for a long time (see, for details, [12]).

Two of the most important and known univalence criteria for analytic functions defined in \mathbb{U}^* were obtained by Becker [1] and Nehari [8]. Some extensions of these two criteria were given by Lewandowski [5], [6] and Ruscheweyh [11]. During the time, unlike there were obtained a lot of univalence criteria by Miazga and Wesolowski [7], Wesolowski [13], Kanas and Srivastava [4] and Deniz and Orhan [2].

In the present paper we consider a general univalence criterion for functions f belonging to the class Σ in terms of the Schwarz derivative defined by

$$S_f(z) = \left(\frac{f''(z)}{f'(z)} \right)' - \frac{1}{2} \left(\frac{f''(z)}{f'(z)} \right)^2.$$

2. LOEWNER CHAINS AND RELATED THEOREM

Before proving our main theorem we need a brief summary of the method of Loewner chains.

Let $\mathcal{L}(z, t) = a_1(t)z + a_2(t)z^2 + \dots$, $a_1(t) \neq 0$, be a function defined on $\mathbb{U} \times [0, \infty)$, where $a_1(t)$ is a complex-valued, locally absolutely continuous function on $[0, \infty)$. $\mathcal{L}(z, t)$ is called a Loewner chain if $\mathcal{L}(z, t)$ satisfies the following conditions;

- (i) $\mathcal{L}(z, t)$ is analytic and univalent in \mathbb{U} for all $t \in [0, \infty)$
- (ii) $\mathcal{L}(z, t) \prec \mathcal{L}(z, s)$ for all $0 \leq t \leq s < \infty$,

where the symbol " \prec " stands for subordination. If $a_1(t) = e^t$ then we say that $\mathcal{L}(z, t)$ is a *standard Loewner chain*.

In order to prove our main results we need the following theorem due to Pommerenke [9] (also see [10]). This theorem is often used to find out univalence for an analytic function, apart from the theory of Loewner chains;

Theorem 2.1. *Let $\mathcal{L}(z, t) = a_1(t)z + a_2(t)z^2 + \dots$ be analytic in \mathbb{U}_r for all $t \in [0, \infty)$. Suppose that;*

- (i) $\mathcal{L}(z, t)$ is a locally absolutely continuous function in the interval $[0, \infty)$, and locally uniformly with respect to \mathbb{U}_r .
- (ii) $a_1(t)$ is a complex valued continuous function on $[0, \infty)$ such that $a_1(t) \neq 0$, $|a_1(t)| \rightarrow \infty$ for $t \rightarrow \infty$ and

$$\left\{ \frac{\mathcal{L}(z, t)}{a_1(t)} \right\}_{t \in [0, \infty)}$$

forms a normal family of functions in \mathbb{U}_r .

- (iii) *There exists an analytic function $p : \mathbb{U} \times [0, \infty) \rightarrow \mathbb{C}$ satisfying $\operatorname{Re} p(z, t) > 0$ for all $z \in \mathbb{U}$, $t \in [0, \infty)$ and*

$$(2.1) \quad z \frac{\partial \mathcal{L}(z, t)}{\partial z} = p(z, t) \frac{\partial \mathcal{L}(z, t)}{\partial t}, \quad z \in \mathbb{U}_r, \quad t \in [0, \infty).$$

Then, for each $t \in [0, \infty)$, the function $\mathcal{L}(z, t)$ has an analytic and univalent extension to the whole disk \mathbb{U} or the function $\mathcal{L}(z, t)$ is a Loewner chain.

The equation (2.1) is called the generalized Loewner differential equation.

3. UNIVALENCE CRITERION FOR THE FUNCTIONS BELONGING TO THE CLASS Σ

In this section, making use of the Theorem 2.1, we obtain an univalence criterion for meromorphic functions. The method of prove is based on Theorem 2.1 and on construction of a suitable Loewner chain.

Theorem 3.1. Let $f, g \in \Sigma$ be locally univalent functions in \mathbb{U}^* . If there exists an analytic function h such that $\operatorname{Re} h(\zeta) \geq \frac{1}{2}$ and $h(\zeta) = 1 + \frac{b_2}{\zeta^2} + \dots$ for $\zeta \in \mathbb{U}^*$, and for arbitrary $\alpha \in \mathbb{C}$ we have

$$(3.1) \quad \left| \frac{1-h(\zeta)}{h(\zeta)} |\zeta|^2 - (|\zeta|^2 - 1) \left[\frac{\zeta h'(\zeta)}{h(\zeta)} + (1-2\alpha) \frac{\zeta f''(\zeta)}{f'(\zeta)} + 2\alpha \frac{\zeta g''(\zeta)}{g'(\zeta)} \right] \right. \\ \left. + \alpha (|\zeta|^2 - 1)^2 \frac{\zeta}{h(\zeta)} \left[\left(\alpha - \frac{1}{2} \right) \left(\frac{f''(\zeta)}{f'(\zeta)} - \frac{g''(\zeta)}{g'(\zeta)} \right)^2 + S_f(\zeta) - S_g(\zeta) \right] \right| \leq 1$$

for all $\zeta \in \mathbb{U}^*$, then f is univalent in \mathbb{U}^* .

Proof. Without loss of generality we can consider the functions of the form

$$f(\zeta) = \zeta + \frac{a_1}{\zeta} + \dots \text{ and } g(\zeta) = \zeta + \frac{b_1}{\zeta} + \dots$$

since the Schwarzian derivative is invariant under Möbius transformations. Consider the functions defined by

$$(3.2) \quad v(\zeta) = \left[\frac{g'(\zeta)}{f'(\zeta)} \right]^\alpha = 1 + \frac{v_2}{\zeta^2} + \dots, \quad \alpha \in \mathbb{C}$$

where we choose this branch of the power $(\cdot)^\alpha$, which for $\zeta \rightarrow \infty$ has value 1, and

$$(3.3) \quad u(\zeta) = f(\zeta)v(\zeta) = \zeta + \frac{u_2}{\zeta} + \dots$$

The functions u and v are meromorphic in \mathbb{U}^* since f and g do not have multiple poles and f' and g' are different from zero.

For all $t \in [0, \infty)$ and $\frac{1}{\zeta} = z \in \mathbb{U}$ the function $f : \mathbb{U}_r \times [0, \infty) \rightarrow \mathbb{C}$ defined formally by

$$(3.4) \quad f(z, t) = \left[\frac{u\left(\frac{e^t}{z}\right) + (e^{-t} - e^t) \frac{1}{z} h\left(\frac{e^t}{z}\right) u'\left(\frac{e^t}{z}\right)}{v\left(\frac{e^t}{z}\right) + (e^{-t} - e^t) \frac{1}{z} h\left(\frac{e^t}{z}\right) v'\left(\frac{e^t}{z}\right)} \right]^{-1} \\ = e^t z + \Psi(e^{-pt}, z^2), \quad p = 1, 2, \dots$$

is analytic in \mathbb{U} since $\Psi(e^{-pt}, z^2)$ is analytic function in \mathbb{U} for each fixed $t \in [0, \infty)$ and $p = 1, 2, \dots$. From (3.4) we have $a_1(t) = e^t$ and

$$\lim_{t \rightarrow \infty} |a_1(t)| = \lim_{t \rightarrow \infty} e^t = \infty.$$

After simple calculation we obtain, for each $z \in \mathbb{U}$,

$$\lim_{t \rightarrow \infty} \frac{f(z, t)}{e^t} = \lim_{t \rightarrow \infty} \left\{ z + \Psi(e^{-(p+1)t}, z^2) \right\} = z.$$

The limit function $k(z) = z$ belongs to the family $\{f(z, t)/e^t : t \in [0, \infty)\}$; then, there exists a number r_0 ($0 < r_0 < 1$) that in every closed disk \mathbb{U}_{r_0} , there exists a constant $K_0 > 0$, such that

$$\left| \frac{f(z, t)}{e^t} \right| < K_0, \quad z \in \mathbb{U}_{r_0}, \quad t \in [0, \infty)$$

uniformly in this disk, provided that t is sufficiently large. Thus, by Montel's Theorem, $\{f(z, t)/e^t\}$ forms a normal family in each disk \mathbb{U}_{r_0} .

Since the function $\Psi(e^{-pt}, z^2)$ is analytic in \mathbb{U} , $\Psi^{(k)}(e^{-pt}, z^2)$ $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$ is continuous on the compact set, so $\Psi^{(k)}(e^{-pt}, z^2)$, $k \in \mathbb{N}_0$ is bounded function. Thus for all fixed $T > 0$, we can write $e^t < e^T$ and we obtain that for all fixed numbers $t \in [0, T] \subset [0, \infty)$, there exists a constant $K_1 > 0$ such that

$$\left| \frac{\partial f(z, t)}{\partial t} \right| < K_1, \quad \forall z \in \mathbb{U}_{r_0}, \quad t \in [0, T].$$

Therefore, the function $f(z, t)$ is locally absolutely continuous in $[0, \infty)$; locally uniformly with respect to \mathbb{U}_{r_0} .

After simple calculations from (3.4) we obtain

$$\begin{aligned} (3.5) \quad & \frac{\partial f(z, t)}{\partial z} \\ &= \frac{1}{z} \frac{e^t}{z} \left\{ \left(1 + (e^{-2t} - 1) \left[h\left(\frac{e^t}{z}\right) + \frac{e^t}{z} h'\left(\frac{e^t}{z}\right) \right] \right) (u'v - v'u) \right. \\ & \quad \left. + (e^{-2t} - 1) \frac{e^t}{z} h\left(\frac{e^t}{z}\right) (u''v - v''u) + (e^{-2t} - 1)^2 \frac{e^{2t}}{z^2} h^2\left(\frac{e^t}{z}\right) (u''v' - v''u') \right\} \\ & \quad \times f^2(z, t) / \left[v\left(\frac{e^t}{z}\right) + (e^{-t} - e^t) \frac{1}{z} h\left(\frac{e^t}{z}\right) v'\left(\frac{e^t}{z}\right) \right]^2 \end{aligned}$$

and

$$\begin{aligned} (3.6) \quad & \frac{\partial f(z, t)}{\partial t} \\ &= -\frac{e^t}{z} \left\{ \left(1 - (e^{-2t} + 1) h\left(\frac{e^t}{z}\right) + (e^{-2t} + 1) \frac{e^t}{z} h'\left(\frac{e^t}{z}\right) \right) (u'v - v'u) \right. \\ & \quad \left. + (e^{-2t} - 1) \frac{e^t}{z} h\left(\frac{e^t}{z}\right) (u''v - v''u) + (e^{-2t} - 1)^2 \frac{e^{2t}}{z^2} h^2\left(\frac{e^t}{z}\right) (u''v' - v''u') \right\} \\ & \quad \times f^2(z, t) / \left[v\left(\frac{e^t}{z}\right) + (e^{-t} - e^t) \frac{1}{z} h\left(\frac{e^t}{z}\right) v'\left(\frac{e^t}{z}\right) \right]^2 \end{aligned}$$

where

$$(3.7) \quad u'v - v'u = f' \left(\frac{g'}{f'} \right)^{2\alpha}, \quad \alpha \in \mathbb{C}$$

$$(3.8) \quad u''v - v''u = (1 - 2\alpha) f'' \left(\frac{g'}{f'} \right)^{2\alpha} + 2\alpha g'' \left(\frac{g'}{f'} \right)^{2\alpha-1}, \quad \alpha \in \mathbb{C}$$

$$(3.9) \quad u''v' - v''u' = \alpha f' \left(\frac{g'}{f'} \right)^{2\alpha} \left\{ (S_f - S_g) + \left(\alpha - \frac{1}{2} \right) \left(\frac{f''}{f'} - \frac{g''}{g'} \right) \right\}, \quad \alpha \in \mathbb{C}$$

and u, v, u', v', u'', v'' are calculated at $\frac{e^t}{z}$.

Consider the function $p : \mathbb{U}_r \times [0, \infty) \rightarrow \mathbb{C}$ for $0 < r < r_0$ and $t \in [0, \infty)$, defined by

$$p(z, t) = z \frac{\partial f(z, t)}{\partial z} \Big/ \frac{\partial f(z, t)}{\partial t}.$$

From (3.5) to (3.9), we can easily see that the function $p(z, t)$ is analytic in \mathbb{U}_r , $0 < r < r_0$. If the function

$$(3.10) \quad w(z, t) = \frac{p(z, t) - 1}{p(z, t) + 1} = \frac{\frac{z \partial f(z, t)}{\partial z} - \frac{\partial f(z, t)}{\partial t}}{\frac{z \partial f(z, t)}{\partial z} + \frac{\partial f(z, t)}{\partial t}}$$

is analytic in $\mathbb{U} \times [0, \infty)$ and $|w(z, t)| < 1$, for all $z \in \mathbb{U}$ and $t \in [0, \infty)$, then $p(z, t)$ has an analytic extension with positive real part ($\operatorname{Re} p(z, t) > 0$) in \mathbb{U} , for all $t \in [0, \infty)$.

To show this we write (3.5) and (3.6) in the equation (3.10), then we obtain

$$(3.11) \quad \begin{aligned} & w(z, t) \\ &= \frac{2 \frac{e^t}{z} \left\{ \left(1 - h \left(\frac{e^t}{z} \right) + (e^{-2t} - 1) \frac{e^t}{z} h' \left(\frac{e^t}{z} \right) \right) (u'v - v'u) \right.}{2e^{-2t} \frac{e^t}{z} h \left(\frac{e^t}{z} \right) (u'v - v'u)} \\ & \quad \left. + \frac{(e^{-2t} - 1) \frac{e^t}{z} h \left(\frac{e^t}{z} \right) (u''v - v''u) + (e^{-2t} - 1)^2 \frac{e^{2t}}{z^2} h^2 \left(\frac{e^t}{z} \right) (u'v' - v''u') \right\}}{2e^{-2t} \frac{e^t}{z} h \left(\frac{e^t}{z} \right) (u'v - v'u)} \\ &= e^{2t} \left(\frac{1 - h \left(\frac{e^t}{z} \right)}{h \left(\frac{e^t}{z} \right)} \right) + (1 - e^{2t}) \frac{e^t}{z} \left(\frac{h' \left(\frac{e^t}{z} \right)}{h \left(\frac{e^t}{z} \right)} + \frac{u''v - v''u}{u'v - v'u} \right) \\ & \quad + e^{2t} (e^{-2t} - 1)^2 \frac{e^{2t}}{z^2} h \left(\frac{e^t}{z} \right) \frac{u'v' - v''u'}{u'v - v'u} \end{aligned}$$

and from (3.7)-(3.9) for all $z \in \mathbb{U}$ and $t \in [0, \infty)$

$$(3.12) \quad \begin{aligned} & w(z, t) \\ &= e^{2t} \left(\frac{1 - h \left(\frac{e^t}{z} \right)}{h \left(\frac{e^t}{z} \right)} \right) + (1 - e^{2t}) \frac{e^t}{z} \left(\frac{h' \left(\frac{e^t}{z} \right)}{h \left(\frac{e^t}{z} \right)} + (1 - 2\alpha) \frac{f'' \left(\frac{e^t}{z} \right)}{f' \left(\frac{e^t}{z} \right)} + 2\alpha \frac{g'' \left(\frac{e^t}{z} \right)}{g' \left(\frac{e^t}{z} \right)} \right) \\ & \quad + \alpha e^{2t} (e^{-2t} - 1)^2 \frac{e^{2t}}{z^2} h \left(\frac{e^t}{z} \right) \left(\left(S_f \left(\frac{e^t}{z} \right) - S_g \left(\frac{e^t}{z} \right) \right) + \left(\alpha - \frac{1}{2} \right) \left(\frac{f'' \left(\frac{e^t}{z} \right)}{f' \left(\frac{e^t}{z} \right)} - \frac{g'' \left(\frac{e^t}{z} \right)}{g' \left(\frac{e^t}{z} \right)} \right) \right). \end{aligned}$$

The right hand side of the equation (3.12) is equal to

$$w(z, 0) = \frac{1 - h \left(\frac{1}{z} \right)}{h \left(\frac{1}{z} \right)}$$

for $t = 0$. Thus, from hypothesis of theorem for $\frac{1}{z} = \zeta \in \mathbb{U}^*$ we have

$$\left| \frac{1 - h(\zeta)}{h(\zeta)} \right| \leq 1.$$

Since $\left| \frac{e^t}{z} \right| \geq |e^t| > 1$ for all $z \in \overline{\mathbb{U}}$ and $t > 0$, we find that $w(z, t)$ is an analytic function in $\overline{\mathbb{U}}$. Then putting $\frac{e^t}{z} = \tilde{\zeta} \in \mathbb{U}^*$, $\tilde{\zeta} = \zeta e^t$, $|\tilde{\zeta}| = e^t$ for $|z| = 1$, from (3.12) by assumption (3.1) replacing $\tilde{\zeta}$ by

ζ we have

$$\begin{aligned}
|w(z, t)| &= \left| |\zeta|^2 \left(\frac{1-h(\zeta)}{h(\zeta)} \right) - (|\zeta|^2 - 1) \left(\frac{\zeta h'(\zeta)}{h(\zeta)} + (1-2\alpha) \frac{\zeta f''(\zeta)}{f'(\zeta)} + 2\alpha \frac{\zeta g''(\zeta)}{g'(\zeta)} \right) \right. \\
&\quad \left. + \alpha (|\zeta|^2 - 1)^2 \frac{e^{2t}}{z^2} h(\zeta) \left((S_f(\zeta) - S_g(\zeta)) + \left(\alpha - \frac{1}{2} \right) \left(\frac{f''(\zeta)}{f'(\zeta)} - \frac{g''(\zeta)}{g'(\zeta)} \right) \right) \right| \\
&\leq 1.
\end{aligned}$$

Therefore $|w(z, t)| < 1$ for all $z \in \mathbb{U}$ and $t \in [0, \infty)$.

Since all the conditions of Theorem 2.1 are satisfied, we obtain that the function $f(z, t)$ is a Loewner chain or has an analytic and univalent extension to the whole unit disk \mathbb{U} , for all $t \in [0, \infty)$.

From (3.2)-(3.4) it follows in particular that

$$f(z, 0) = \frac{v(\frac{1}{z})}{u(\frac{1}{z})} = \frac{1}{f(\frac{1}{z})} \in \mathcal{S}$$

and for $\frac{1}{z} = \zeta \in \mathbb{U}^*$ we say that $f(\zeta)$ is univalent in \mathbb{U}^* . Thus the proof is completed. \square

For $\alpha = 0$ in Theorem 3.1 we obtain following new result:

Corollary 3.2. *Let $f \in \Sigma$ be locally univalent function in \mathbb{U}^* . If there exists an analytic function h with $\operatorname{Re} h(\zeta) \geq \frac{1}{2}$ in \mathbb{U}^* and $h(\zeta) = 1 + \frac{h_2}{\zeta^2} + \dots$ such that*

$$(3.13) \quad \left| \frac{1-h(\zeta)}{h(\zeta)} |\zeta|^2 - (|\zeta|^2 - 1) \left[\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta f''(\zeta)}{f'(\zeta)} \right] \right| \leq 1$$

for all $\zeta \in \mathbb{U}^*$, then f is univalent in \mathbb{U}^* .

For $\alpha = \frac{1}{2}$ in Theorem 3.1 we obtain univalence criterion given by Miazga and Wesolowski [7].

Corollary 3.3. *Let $f, g \in \Sigma$ be locally univalent functions in \mathbb{U}^* . If there exists an analytic function h with $\operatorname{Re} h(\zeta) \geq \frac{1}{2}$ in \mathbb{U}^* and $h(\zeta) = 1 + \frac{h_2}{\zeta^2} + \dots$ such that*

$$\begin{aligned}
(3.14) \quad &\left| \frac{1-h(\zeta)}{h(\zeta)} |\zeta|^2 - (|\zeta|^2 - 1) \left[\frac{\zeta h'(\zeta)}{h(\zeta)} + \frac{\zeta g''(\zeta)}{g'(\zeta)} \right] \right. \\
&\quad \left. + \frac{1}{2} (|\zeta|^2 - 1)^2 \frac{\zeta}{h(\zeta)} [(S_f(\zeta) - S_g(\zeta))] \right| \leq 1
\end{aligned}$$

for all $\zeta \in \mathbb{U}^*$, then f is univalent in \mathbb{U}^* .

For $h(\zeta) = 1$ and $\alpha = \frac{1}{2}$ in Theorem 3.1 we obtain sufficient condition of Epstein type [3] on the exterior of the unit disk obtained earlier by Wesolowski [13].

Corollary 3.4. *Let $f, g \in \Sigma$ be locally univalent functions in \mathbb{U}^* . If the following inequality*

$$(3.15) \quad \left| \frac{1}{2}(|\zeta|^2 - 1)^2 \frac{\zeta}{\bar{\zeta}} [(S_f(\zeta) - S_g(\zeta))] - (|\zeta|^2 - 1) \frac{\zeta g''(\zeta)}{g'(\zeta)} \right| \leq 1$$

is satisfied for all $\zeta \in \mathbb{U}^$, then f is univalent in \mathbb{U}^* .*

For $f(\zeta) = g(\zeta)$, $h(\zeta) = 1$ and $\alpha = \frac{1}{2}$ in Theorem 3.1 we obtain well-known Becker's univalence criterion [1] in \mathbb{U}^* .

Corollary 3.5. *Let $f \in \Sigma$ be locally univalent function in \mathbb{U}^* . If the following inequality*

$$(3.16) \quad (|\zeta|^2 - 1) \left| \frac{\zeta f''(\zeta)}{f'(\zeta)} \right| \leq 1$$

is satisfied for all $\zeta \in \mathbb{U}^$, then f is univalent in \mathbb{U}^* .*

For $g(\zeta) = \zeta$, $h(\zeta) = 1$ and $\alpha = \frac{1}{2}$ in Theorem 3.1 we obtain Nehari type univalence criterion [8] in \mathbb{U}^* .

Corollary 3.6. *Let $f \in \Sigma$ be locally univalent function in \mathbb{U}^* . If the following inequality*

$$(3.17) \quad |S_f(\zeta)| \leq \frac{2}{(|\zeta|^2 - 1)^2}$$

is satisfied for all $\zeta \in \mathbb{U}^$, then f is univalent in \mathbb{U}^* .*

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